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OPTIMAL CONTROL OF THE WORKLOAD IN AN M/G/1 QUEUEING SYSTEM WITH REMOVABLE SERVER

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Optimal Control of the Workload in an M/G/1 Queueing System with Removable Server

by

H.C. Tijms

ABSTRACT

This paper considers an M/G/1 queueing system in which the server may be either turned on or turned off. The system can be controlled at the arrival epochs by taking an action to change the current position of the server or not. The control is based on the total amount of work remaining to be done in the system. The costs consist of a linear holding cost for the workload and a fixed cost for turning the server on. In this paper it will be shown that among a wide class of policies there is an average cost optimal policy which belongs to the subclass of the so-called D-policies where a D-policy is a stationary policy that turns the server off only when the system is empty and turns the server on only when the workload exceeds the level D. Also, an alternative derivation of the formula for the average cost of a D-policy will be given.

KEY WORDS & PHRASES: *Optimal control, workload, M/G/1 queue, removable server, D-policy, average cost optimal*

1. INTRODUCTION

We consider an M/G/1 queueing system with removable server where jobs arrive in accordance with a Poisson process with rate λ. Each job involves an amount of work which is known upon arrival of the job. The amounts of work of the jobs are independently sampled from a distribution having probability distribution function $F(\cdot)$ with finite first moment μ and finite second moment $\mu^{(2)}$. It is assumed that $F(0) = 0$ and $\rho < 1$ where $\rho = \lambda\mu$. Each job will be serviced. The service mechanism may be either turned on or turned off, where service will be provided only when the server is on. The server will be off when the system is empty. The control of the system will be based on the workload, where the workload at time t is defined as the total amount of work remaining to be done in the system at time t (i.e. the workload at time t is the virtual waiting time at time t). The following costs are considered. There is a fixed cost of $K > 0$ for turning the server on and there is a holding cost of $h > 0$ per unit worload per unit time.

We shall assume that the system can be only controlled at the arrival epochs of the jobs. At these epochs an action which is based on the workload must be taken to change the current position of the server or not. A policy π for controlling the system is any measurable rule for choosing actions at the arrival epochs, where a policy is said to be stationary if the action it chooses only depends on the current state of the system. We are interested in a policy that minimizes the average cost.

In this paper it will be shown that among a wide class of policies there is an average cost optimal policy which belongs to the subclass of the so-called D-policies. A D-policy is a stationary policy which turns the server off only when the system is empty and turns the server on when the workload exceeds the value D where D is a non-negative number. The D-policy was studied in BALACHANDRAN [1] and BALACHANDRAN & TIJMS [2] where a formula for the average cost of the D-policy was derived by using a standard result from the theory of regenerative processes. Related work was done in BELL [3] and HEYMAN [7] where the queue size was taken as control variable and the average cost optimality of a policy similiar to the D-policy was established.

In section 2 of this paper we give some preliminaries. Also, in this section we present a short alternative derivation of the formula for the average cost of the D-policy. This derivation in itself may be of interest. In section 3 we establish the optimality equation for the average cost criterion and from this equation the existence of an average cost optimal policy which is of the D-type will be derived.

2. PRELIMINARIES

We first introduce the functions $t(x)$ and $h(x)$ which will play an important role in our considerations. For the queueing system in which the server is always on when the system is not empty and the workload at epoch 0 equals x, let $t(x)$ be the expectation of the first epoch at which the system is empty and let $h(x)$ be the expected holding cost incurred up to that epoch. Using a standard argument from busy period analysis, it is easy to prove the following well known results (e.g. TIJMS [12])

$$(1) \qquad t(x) = \frac{x}{1-\rho} \text{ and } h(x) = \frac{hx^2}{2(1-\rho)} + \frac{h\lambda\mu^{(2)}x}{2(1-\rho)^2} \qquad \text{for all } x \geq 0.$$

To define the average cost criterion, we first observe that at each point of time the state of the system can be described by a point in the set

$$S = \{x | x \text{ real}, x \geq 0\} \cup \{x' | x \text{ real}, x > 0\},$$

where state $x(x')$ corresponds to the situation where the workload equals x and the server is off (on). Let $X(t)$ be the state of the system at time t, where the process $\{X(t)\}$ is assumed to be continuous from the right. Further, let $Z(t)$ be the total cost incurred during $(0,t]$. Define $\tau_0 = 0$ and, for $n \geq 1$, let τ_n be the arrival epoch of the nth job. For any $n \geq 1$, let Z_n be the total cost incurred during $(\sum_{k=0}^{n-1} \tau_k, \sum_{k=0}^{n} \tau_k]$ where the cost of $K > 0$ is included in Z_n when the server is turned on at epoch $\sum_{k=0}^{n} \tau_k$.

For any policy π, define *)

*) The results of this paper also apply when we replace in (2) limsup by liminf.

(2) $$\phi(s,\pi) = \limsup_{n\to\infty} E_{s,\pi}(\sum_{k=1}^{n} Z_k)/E_{s,\pi}(\sum_{k=1}^{n} \tau_k) =$$

$$= \limsup_{n\to\infty} \frac{\lambda}{n} E_{s,\pi}(\sum_{k=1}^{n} Z_k) \qquad \text{for } s \in S,$$

where $E_{s,\pi}$ denotes the expectation when the initial state is s and policy π is used. For any class C of policies, a policy $\pi^* \in C$ is said to be average cost optimal among the class C of policies when $\phi(s,\pi^*) \le \phi(s,\pi)$ for all $s \in S$ and $\pi \in C$. For any positive number M, we define C_M as the class of policies under which the server will be always on when the workload is larger than M. Observe that $C_M \supset C_K$ when $M > K$.

REMARK 1. A criterion which is more natural than (2) is given by

$$\psi(s,\pi) = \limsup_{t\to\infty} t^{-1}E_{s,\pi}(Z(t)).$$

However, to derive from the optimality equation for the average cost criterion the existence of an average cost optimal policy we have to deal with (2). Nevertheless for certain classes of policies the criteria ϕ and ψ are identical to each other. Let C_{Ms} be the class of all stationary policies which belong to C_M and never turn the server off when the system is not empty. Observe that the D-policy belongs to C_{Ds}. Then, for each $\pi \in C_{Ms}$, we have by (1) that both $E_{s,\pi}(T)$ and $E_{s,\pi}(Z(T))$ are finite for all $s \in S$, where T is the epoch of the first return of the system to state 0. Now, by Theorem 1 of ROSS [10], for any $\pi \in C_{Ms}$,

(3) $$\phi(s,\pi) = \psi(s,\pi) = E_{0,\pi}(Z(T))/E_{0,\pi}(T) \qquad \text{for all } s \in S.$$

We shall now give an alternative derivation of the formula for the average costs of the D-policy. This derivation will use a properly chosen Markov process embedded in the process $\{X(t)\}$. Fix $D \ge 0$ and consider the queueing system controlled by the D-policy. Since the average cost of the D-policy is independent of the initial state, we assume for convenience that the system is empty at epoch 0. We now consider a Markov process embedded at the epochs where either the system becomes empty or a job arrives and the server is left off. To be precise, let $T_0 = 0$ and, for $n \geq 1$, let T_n be the nth epoch such that $X(T_n) \in \{x | 0 \leq x \leq D\}$ and $X(T_n) \neq X(T_n^-)$. For any $n \geq 0$, let $Y_n = X(T_n)$. Then, the discrete-time process $\{Y_n\}$ is a Markov chain having the continuous state space $[0,D]$. Let $P_x^n(y) = \Pr\{Y_{n+1} \leq y | Y_1 = x\}$ for $n \geq 1$, where we write $P_x^1(y) = P_x(y)$. It is immediate to verify that $\{Y_n\}$ has the so-called uniform ϕ-recurrence property (see p.26 in OREY [9] and choose a ϕ-measure such that $\phi(A) > 0$ if and only if $0 \in A$). Now, by Theorem 7.1 of OREY [9], the Markov chain $\{Y_n\}$ has a unique stationary probability distribution function $Q(y)$, $0 \leq y \leq D$, such that

$$(4) \qquad Q(y) = \int_0^D P_x(y)dQ(x) \qquad \text{for all } 0 \leq y \leq D$$

and

$$(5) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_x^k(y) = Q(y) \qquad \text{for all } 0 \leq x,y \leq D. \ ^{*)}$$

The stationary distribution Q can be explicitly given. From (4),

$$Q(y) = \int_0^D \{F(y-x) + 1 - F(D-x)\}dQ(x) =$$

$$= \int_0^D \{1 - F(D-x)\}dQ(x) + \int_0^y Q(y-x)dF(x) \quad \text{for all } 0 \leq y \leq D$$

from which we get $Q(y) = Q(0) + \int_0^y Q(y-x)dF(x)$ for $0 \leq y \leq D$. It follows from this renewal equation that $Q(y) = Q(0)\{1 + M(y)\}$ where $M(y) = \sum_{n=1}^{\infty} F^n(y)$ with F^n is the n-fold convolution of F with itself. Together this and $Q(D) = 1$ show

$$(6) \qquad Q(y) = \{1 + M(y)\}/\{1 + M(D)\} \qquad \text{for all } 0 \leq y \leq D.$$

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*) The results (4) and (5) can also be established by observing that for the Markov chain $\{Y_n\}$ the main recurrence time from state x to state 0 is finite for all x and following a reasoning similiar to that on p.365 in FELLER [6].

Now, for any $n \geq 1$, let $\bar{Z}_n$ be the total cost incurred during $(T_{n-1}, T_n]$ and let $\bar{\tau}_n = T_n - T_{n-1}$. Then, by the proof of Theorem 1 of ROSS [10], we have

$$\lim_{t\to\infty} \frac{1}{t} EZ(t) = \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^{n} \bar{Z}_k\right) \Big/ \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^{n} \bar{\tau}_k\right).$$

Hence, by (5), the average cost of the D-policy to be denoted by g(D) equals

$$g(D) = \int_0^D \bar{Z}(x)dQ(x) \Big/ \int_0^D \bar{\tau}(x)dQ(x) \tag{7}$$

where $\bar{Z}(x) = E(\bar{Z}_k | Y_k = x)$ and $\bar{\tau}(x) = E(\bar{\tau}_k | Y_k = x)$, $0 \leq x \leq D$. Now, by the definitions of the functions $\bar{Z}(x)$ and $h(x)$, we have for all $0 \leq x \leq D$,

$$\bar{Z}(x) = \frac{hx}{\lambda} + \int_{D-x}^{\infty} \{K + h(x+y)\}dF(y) = K + k(x) - \int_0^{D-x} \{K + h(x+y)\}dF(y) \tag{8}$$

where

$$k(x) = \frac{hx}{\lambda} + \int_0^{\infty} h(x+y)dF(y) = h(x) + \frac{hx}{\lambda(1-\rho)} + \frac{h\mu^{(2)}}{2(1-\rho)^2} \quad \text{for } x \geq 0. \tag{9}$$

Now, let $\delta(x) = 1$ for $x > 0$ and let $\delta(0) = 0$, then we can write (8) as

$$\bar{Z}(x) = K + k(x) - \int_0^D \{K\delta(u) + h(u)\}dP_x(u) \qquad \text{for } 0 \leq x \leq D. \tag{10}$$

By integrating both sides of (9) with respect to Q and using (4), we get after an interchange of the order of integration

$$\int_0^D \bar{Z}(x)dQ(x) = KQ(0) + \int_0^D \{k(x) - h(x)\}dQ(x). \tag{11}$$

In the same way, we get from $\bar{\tau}(x) = 1/\lambda + \int_{D-x}^{\infty} t(x+y)dF(y)$,

$$\int_0^D \bar{\tau}(x)Q(dx) = \int_0^D \{s(x) - t(x)\}dQ(x) \tag{12}$$

where

$$s(x) = \frac{1}{\lambda} + \int_0^{\infty} t(x+y)dF(y) = t(x) + \frac{1}{\lambda(1-\rho)} \qquad \text{for } x \geq 0. \tag{13}$$

Now, by (1) and (6)-(13),

$$(14)\qquad g(D) = \frac{K\lambda(1-\rho)+h\int_0^D y\,dM(y)}{1+M(D)} + \frac{h\lambda\mu^{(2)}}{2(1-\rho)}$$

$$= \frac{K\lambda(1-\rho)-h\{D+\int_0^D M(y)dy\}}{1+M(D)} + hD + \frac{h\lambda\mu^{(2)}}{2(1-\rho)}\ .$$

It is easily verified that the function $g(D)$, $D \geq 0$, is minimal for the unique positive D^* (say) satisfying

$$(15)\qquad D^* + \int_0^{D^*} M(x)dx = \frac{K\lambda(1-\rho)}{h}\ .$$

Observe that, by (14) and (15),

$$(16)\qquad g(D^*) = hD^* + \frac{h\lambda\mu^{(2)}}{2(1-\rho)}\ .$$

To end this section, we give the following lemma which is due to Professor J.W. Cohen (private communication).

LEMMA 1. *For the queueing system where the server is always on when the system is not empty and the jobs are serviced in order of arrival, let* N_n *be the number of jobs present just after the service completion epoch of the* nth *job. Then* $\lim_{n\to\infty} n^{-1}EN_n^2 = 0$.

This lemma is easily verified by working out the obvious identity

$$\frac{1}{n}\sum_{k=1}^{n} EN_{k+1}^2 = \frac{1}{n}\sum_{k=1}^{n} E\{N_k - \delta(N_k) + \nu_{k+1}\}^2 \quad \text{for all } n \geq 1,$$

where ν_{k+1} denotes the number of jobs arriving during the service time of the (k+1)th job, and by using the fact that $\lim_{n\to\infty} EN_n = \rho + \lambda^2\mu^{(2)}/2(1-\rho)$ and $\lim_{n\to\infty} E\delta(N_n) = \rho$ (e.g. COHEN [4]). Clearly, Lemma 1 implies that for the queueing system in which the server is always on when the system is not empty holds

(17) $$\lim_{n\to\infty} \frac{EW_n^j}{n} = 0 \qquad \text{for } j = 1,2,$$

where W_n denotes the workload at the arrival epoch of the nth job.

3. THE AVERAGE COST OPTIMALITY OF THE D^*-POLICY

In this section we shall first prove that the optimality equation for the average cost criterion applies to the model considered in this paper. The solution of this equation will be explicitly given. Using the optimality equation it will be next shown that the D^*-policy is average cost optimal among the class C_M of policies for each M.

To shorten the notation, we introduce the actions 0 and 1 where action 0 in state $x(x')$ prescribes to leave the server off (on) and action 1 in state $x(x')$ prescribes to turn the server on (off). Denote by $c(s,a)$ the expected cost incurred until the next arrival of a job when action a is taken in state s. Then, for all $x \geq 0$,

$$c(x,0) = \frac{hx}{\lambda}, \quad c(x',0) = e^{-\lambda x}\frac{hx^2}{2} + h\int_0^x (xt-\tfrac{1}{2}t^2)\lambda e^{-\lambda t}dt$$

$$c(x,1) = K + c(x',0) \text{ and } c(x',1) = c(x,0).$$

Now, let D^* be the unique positive number satisfying (15) and let

(18) $$g^* = g(D^*).$$

To prove that the D^*-policy is not only average cost optimal among the subclass of the D-policies but also among the class C_M of policies, we define the function $v(s)$, $s \in S$, as follows

(19) $$v(x) = \begin{cases} K + h(x) - g^*t(x) & \text{for } x > D^* \\ \frac{hx}{\lambda} - \frac{g^*}{\lambda} + \int_0^\infty v(x+y)dF(y) & \text{for } 0 \leq x \leq D^* \end{cases}$$

and

$$(20) \qquad v(x') = h(x) - g^{*}t(x) \qquad \text{for } x > 0.$$

The function v(s) is uniquely determined by this definition and is finite. To prove this, we observe that the second part of (19) can be written as

$$(21) \qquad v(x) = \alpha(x) + \int_{0}^{D^{*}-x} v(x+y)dF(y) \qquad \text{for } 0 \le x \le D^{*}$$

where

$$(22) \qquad \alpha(x) = \frac{hx}{\lambda} - \frac{g^{*}}{\lambda} + \int_{D^{*}-x}^{\infty} \{K+h(x+y)-g^{*}t(x+y)\}dF(y) \quad \text{for } 0 \le x \le D^{*}.$$

By introducing the new variable $u = D^{*} - x$ the equation (21) is easily converted into the standard form of the renewal equation. This shows that v(x) is uniquely determined and given by

$$(23) \qquad v(x) = \alpha(x) + \int_{0}^{D^{*}-x} \alpha(x+y)dM(y) \qquad \text{for } 0 \le x \le D^{*}.$$

It easily follows from the definitions of the function h(x), t(x) and v(s) and the uniqueness of the solution to (19) that v(s) can be interpreted as follows

(24) v(s) = the expected cost incurred until the first return of the system to state 0 minus g^{*} times the expected time until the first return of the system to state 0 when the initial state is s and the D^{*}-policy is used.

By a standard result in the theory of the regenerative processes we have for the D^{*}-policy that $g(D^{*})$ equals the ratio of the expected cost incurred between two successive returns to state 0 and the expected time between two successive returns to state 0 (cf. (3)). Hence, by (18), (19) and (24), we have v(0) = 0 which shows that

$$(25) \qquad \int_0^\infty v(y)dF(y) = \frac{g^*}{\lambda}\,.$$

We shall now prove that $\{g^*,v(s)\}$ satisfies the optimality equation for the average cost criterion (it might be interesting to note that the definition of v(s) was inspired by results in DERMAN & VEINOTT [5]). To do this, we first establish a number of properties of the function v(s).

<u>LEMMA 2</u>.

(a) $c(x',0) - g^*/\lambda + e^{-\lambda x}\int_0^\infty v(y)dF(y) + \int_0^x \lambda e^{-\lambda t}dt \int_0^\infty v((x-t+y)')dF(y) =$
$= h(x) - g^*t(x)$ *for all* $x > 0$.

(b) $hx/\lambda - g^*/\lambda + \int_0^\infty v(x+y)dF(y) = K + h(x) - g^*t(x) + (hx-hD^*)/\lambda(1-\rho)$ *for all* $x \geq D^*$.

(c) $h(x) - g^*t(x) \leq hx/\lambda - g^*/\lambda + \int_0^\infty v(x+y)dF(y) \leq K + h(x) - g^*t(x)$ *for all* $0 \leq x \leq D^*$.

<u>PROOF</u>.

(a) en (b). These assertions follow from (1), (16), (18), (19) and (25) after some straightforward calculations.

(c) By assertion (b) and (10), we get from (22) that, for all $0 \leq x \leq D^*$,
$\alpha(x) = K + h(x) - g^*t(x) + \frac{hx-hD^*}{\lambda(1-\rho)} - \int_0^{D^*-x}\{K + h(x+y) - g^*t(x+y)\}dF(y)$.
Using this and the renewal equation $M(y) = F(y) + \int_0^y F(y-v)dM(v)$ for $y \geq 0$, we get from (23) after some manipulations with integrals that, for $0 \leq x \leq D^*$,

$$v(x) = K + h(x) - g^*t(x) - \frac{h}{\lambda(1-p)}\{D^* - x + \int_0^{D^*-x} M(y)dy\}.$$

Hence, for all $0 \leq x \leq D^*$,

$$K + h(x) - g^*t(x) - v(x) \geq 0.$$

Further, using (15), we have for all $0 \leq x \leq D^*$

$$h(x) - g^*t(x) - v(x) \leq - K + \frac{h}{\lambda(1-\rho)}\{D^* + \int_0^{D^*} M(y)dy\} = 0.$$

Together these inequalities and definition (19) prove assertion (c).

We are now in a position to prove that $\{g^*, v(s)\}$ satisfies the optimality equation for the average cost criterion.

THEOREM 1.

$$v(x) = \min\{c(x,0) - \frac{g^*}{\lambda} + e^{-\lambda x}\int_0^\infty v(y)dF(y) + \int_0^x \lambda e^{-\lambda t}dt \int_0^\infty v((x-t+y')dF(y),$$

$$c(x',1) - \frac{g^*}{\lambda} + v(x+y)dF(y)\}, \qquad x > 0,$$

where $v(x)$ *equals the first (second) expression between brackets for all* $0 \le x \le D^*$ *(*$x > D^*$*). Further,*

$$v(x') = \min\{c(x',0) - \frac{g^*}{\lambda} + e^{-\lambda x}\int_0^\infty v(y)dF(y) + \int_0^x \lambda e^{-\lambda t}dt \int_0^\infty v((x-t+y')dF(y),$$

$$c(x',1) - \frac{g^*}{\lambda} + \int_0^\infty v(x+y)dF(y)\}, \qquad x > 0,$$

where $v(x')$ *equals the first expression between brackets for all* $x > 0$.

PROOF. By assertion (a) of Lemma 2, it suffices to prove that

$$v(x) = \min\{\frac{hx}{\lambda} - \frac{g^*}{\lambda} + \int_0^\infty v(x+y)dF(y),\ K + h(x) - g^*t(x)\}, \qquad x \ge 0$$

and

$$v(x') = \min\{h(x) - g^*t(x),\ \frac{hx}{\lambda} - \frac{g^*}{\lambda} + \int_0^\infty v(x+y)dF(y)\}, \qquad x > 0.$$

The Theorem now follows immediately from the assertion (b) and (e) of Lemma 2 and the definitions (19) and (20).

Finally, we prove the average cost optimality of the D^*-policy.

THEOREM 2. *For each* $M \ge 0$, *the* D^**-policy is average cost optimal among the class* C_M *of policies.*

PROOF. Since $C_M \supset C_K$ for $M > K$, it is no restriction to consider only the classes C_M with $M \geq D^*$. Fix now $M \geq D^*$ and consider the class C_M. The proof of Theorem 2 in ROSS [10] shows that Theorem 2 follows from Theorem 1 when

$$(26) \qquad \lim_{n\to\infty} \frac{1}{n} E_{s,\pi}[v(X_n)] = 0 \qquad \text{for all } s \in S \text{ and } \pi \in C_M$$

where X_n denotes the state of the system at the arrival epoch of the nth job. To prove (26), we first observe that by the definition of the function $v(s)$ and (1) there are finite numbers γ_1, γ_2 and γ_3 such that, for all $x > 0$,

$$(27) \qquad |v(x)| \leq \gamma_1 x^2 + \gamma_2 x + \gamma_3 \text{ and } |v(x')| \leq \gamma_1 x^2 + \gamma_2 x + \gamma_3.$$

Further, for any initial state $s \in S$ and policy $\pi \in C_M$, we have with probability 1 that $X_n \leq W_n$ where W_n denotes the workload at the arrival epoch of the nth job for the queueing system in which the server is always on when the system is not empty. Now, (26) follows from (17) and (27). □

REMARK 2. In this paper we have assumed that the workload could be only controlled at the arrival epochs. It seems reasonable to conjecture that the D^*-policy is also average cost optimal among the class of controls which allow for controlling the workload at each point of time.

A more general problem is that in which the workload can be controlled by using one of two service rates where an amount of work σ_i will be processed per unit time when service rate i is used with $0 \leq \sigma_1 < \sigma_2$. This paper assumed $\sigma_1 = 0$. For the case of $\sigma_1 > 0$ with $\lambda\mu/\sigma_1 < 1$, a formula for the average cost of the so-called (y_1,y_2) policy has been derived in TIJMS [12], where an (y_1,y_2) policy is a stationary policy which switches from service rate 1 to service rate 2 only when the workload exceeds the level y_1 and switches from service rate 2 to service rate 1 when the workload falls to the level y_2 with $y_2 \leq y_1$. Also, for this problem it might be conjectured that there is an average cost optimal policy which is of the (y_1,y_2) type. Such an optimality question was studied in MITCHELL [8] and in THATCHER [11] under the assumption that there is no cost for changing the service rate.

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